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LETTER TO THE EDITOR

Analytical expressions for the matrix elements of the non-compact symplectic algebra

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Abstract. Analytic matrix elements are derived for all the lowest weight representations of the non-compact symplectic algebra sp(3, R). It is shown that the expressions are exact for representations of the type ($\sigma_1 = \sigma_2 = \sigma_3$) and for all states of an arbitrary representation that are multiplicity free with respect to the u(3) subalgebra. Furthermore they are remarkably accurate in general.

The symplectic group Sp(3, R) is best known as the dynamical group of the three-dimensional harmonic oscillator (Wybourne 1974). However, it is also the dynamical group of a nuclear collective model (Rosensteel and Rowe 1977a). The representations that occur in the latter application are the lowest weight representations (Godement 1958, Rosensteel and Rowe 1977b) indexed by a triple of integers or semi-integers, (σ_1, σ_2, σ_3). In the past, matrix elements of the sp(3, R) algebra were calculated numerically for these representations (Rosensteel 1980, Rosensteel and Rowe 1983). Recently Castaños *et al* (1983) gave analytic expressions for the case ($\sigma_1 = \sigma_2 = \sigma_3$). In this letter, we show that their results are a special case of a general analytic expression that gives exact matrix elements whenever the states involved are multiplicity free. This includes all the states of a ($\alpha_1 = \sigma_2 = \sigma_3$) representation and many states for a generic representation. It is further demonstrated that the analytic expression is remarkably accurate even for states with multiplicity, and is more than adequate, for practical purposes, for the larger σ representations that occur in nuclear physics applications (Park *et al* 1984) for example. However, when greater accuracy is required, the analytic expression provides a good first approximation for a precise numerical calculation (Rosensteel and Rowe 1983). The coherent state equations can also be solved exactly (Rowe 1983).

Except for the symplectic Lie algebras, analytic formulae for the matrix elements of the classical Lie algebras have been known for some time. In particular, the classic work of Gel'fand and Tseitlin (1950a, b) gave formulae for the finite-dimensional representations of the unitary and orthogonal Lie algebras and, by analytic continuation (Gel'fand and Graev 1965, Lemire and Patera 1979), of their non-compact forms. Although we consider only sp(3, R) in this letter, our technique easily extends to arbitrary sp(n, R) and has the advantage that it explicitly separates out the u(n) isoscalar factor (Baird and Biedenharn 1963).

|| On leave from the University of Toronto.

The strategy is to use coherent state theory (Bargmann 1972) to relate the sp(3, R) algebra to a simpler u(3)-boson algebra, whose matrix elements are known (Rosensteel and Rowe 1983). A formulation of the coherent state theory of Sp(3, R) is presented in Rowe (1983).

A familiar realisation of the sp(3, R) algebra (Rosensteel and Rowe 1977a) is given by the basis $A_{ij} = \Sigma_n b^\dagger_{ni} b^\dagger_{nj}$, $B_{ij} = \Sigma_n b_{ni} b_{nj}$, $C_{ij} = \frac{1}{2}\Sigma_n (b^\dagger_{ni} b_{nj} + b_{nj} b^\dagger_{ni})$, where $i, j = 1, 2, 3$ and $(b^\dagger_{ni}, b_{ni})$ are Weyl boson operators. From the boson commutators $[b_{ni}, b^\dagger_{mj}] = \delta_{mn}\delta_{ij}$ one readily infers the sp(3, R) structure. A coherent state realisation of this basis (Rowe 1983) is given by

$$\Gamma(A_{ij}) = (\mathbb{C}z)_{ij} + (\mathbb{C}z)_{ji} - 4z_{ij} + (z\nabla z)_{ij},$$
$$\Gamma(B_{ij}) = \nabla_{ij}, \qquad \Gamma(C_{ij}) = \mathbb{C}_{ij} + (z\nabla)_{ij}, \tag{1}$$

where (z_{ij}) is a symmetric 3×3 array of six linearly independent complex variables, $\nabla_{ij} = (1+\delta_{ij})\partial/\partial z_{ij}$ and $(\mathbb{C}_{ij})$ are a basis for an 'intrinsic' u(3) algebra with $[\mathbb{C}_{ij}, z_{lk}] = [\mathbb{C}_{ij}, \nabla_{lk}] = 0$. Note that we use matrix notation so that, for example, $(z\nabla)_{ij} = \Sigma_k z_{ik}\nabla_{kj}$. One can readily ascertain that $[\Gamma(X), \Gamma(Y)] = \Gamma([X, Y])$, confirming that Γ is a realisation.

Now it has been shown by Rosensteel and Rowe (1982a, b) that, as $\sigma \to \infty$, the sp(3, R) algebra contracts to a u(3)-boson algebra

$$A_{ij} \to \sqrt{2\sigma}\, a^\dagger_{ij}, \qquad B_{ij} \to \sqrt{2\sigma}\, a_{ij}, \qquad C_{ij} \to \mathbb{C}_{ij} + (a^\dagger a)_{ij}, \tag{2}$$

where $a^\dagger_{ij}$ and a_{ij} are Weyl boson operators satisfying

$$[a_{ij}, a^\dagger_{lk}] = \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl},$$
$$[\mathbb{C}_{ij}, a^\dagger_{lk}] = [\mathbb{C}_{ij}, a_{lk}] = 0, \tag{3}$$

and σ is the value of the intrinsic u(1) operator $\frac{1}{3}\operatorname{Tr}\mathbb{C}$.

The coherent state realisation of the u(3)-boson algebra, parallel to equation (1),

$$\gamma(a^\dagger_{ij}) = z_{ij}, \qquad \gamma(a_{ij}) = \nabla_{ij}, \qquad \gamma(C_{ij}) = \mathbb{C}_{ij} + (z\nabla)_{ij}, \tag{4}$$

reveals the very close relationship between the two algebras.

Evidently the realisation Γ of sp(3, R) defines an action of the sp(3, R) algebra on u(3)-boson coherent state wavefunctions. However, since the Hermitian adjoint of ∇_{ij} is z_{ij}, with respect to the u(3)-boson measure, it is evident that Γ is not unitary; i.e. $\Gamma(A_{ij})^\dagger \neq \Gamma(B_{ij})$. To obtain a unitary realisation, we therefore seek a transformation

$$\gamma(X) = \kappa^{-1}\Gamma(X)\kappa, \qquad X \in \mathrm{sp}(3, R), \tag{5}$$

with $\kappa = \kappa^\dagger$ a u(3) scalar. The requirement of unitarity, $\gamma(B_{ij})^\dagger = \gamma(A_{ij})$, then implies

$$\Gamma(A_{ij}) = \kappa^2 z_{ij} \kappa^{-2}. \tag{6}$$

Now, one can show, by direct substitution, that

$$\Gamma(A_{ij}) = [\Lambda, z_{ij}], \tag{7}$$

where Λ is the u(3) scalar operator

$$\Lambda = \tfrac{1}{2}\operatorname{Tr}[(\boldsymbol{C} + z\nabla)(\boldsymbol{C} + z\nabla)] + \tfrac{1}{3}\operatorname{Tr}(\mathbb{C})\operatorname{Tr}(z\nabla) - \tfrac{1}{4}\operatorname{Tr}(z\nabla z\nabla) - \operatorname{Tr}(z\nabla), \tag{8}$$
$$\boldsymbol{C} = \mathbb{C} - \tfrac{1}{3}\operatorname{Tr}(\mathbb{C}).$$

Thus we obtain, from equation (6), the equation for κ

$$[\Lambda, z_{ij}] = \kappa^2 z_{ij} \kappa^{-2}. \tag{9}$$

We also obtain the manifestly unitary expression of γ for sp(3, R)

$$\gamma(A_{ij}) = \kappa z_{ij} \kappa^{-1}, \qquad \gamma(B_{ij}) = \kappa^{-1} \nabla_{ij} \kappa, \tag{10}$$
$$\gamma(C_{ij}) = \mathbb{C}_{ij} + (z\nabla)_{ij}.$$

To solve equation (9) for κ, we consider first the matrix elements of z_{ij} and Λ. A u(3)-boson representation is characterised by the U(3) quantum numbers $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of its boson vacuum state $|\sigma)$ (Rosensteel and Rowe 1982a, b). A basis for the representation is constructed by first combining boson raising operators into u(3) tensors of rank $n = (n_1, n_2, n_3)$ and then coupling the tensors to the boson vacuum state to form an orthonormal basis of states $|\sigma n \rho \omega)$ of total u(3) symmetry $\omega = (\omega_1, \omega_2, \omega_3)$ with multiplicity ρ. Matrix elements of the boson operators in this basis are given by Rosensteel and Rowe (1983),

$$\begin{aligned}(\sigma n' \rho' \omega' \| a^\dagger \| \sigma n \rho \omega) \\ = (-1)^{\lambda' + \mu' - \lambda - \mu} \times (n' \| a^\dagger \| n) \times \mathrm{U}((\lambda_\sigma \mu_\sigma)(\lambda_n \mu_n)(\lambda' \mu')(20); (\lambda\mu)\rho(\lambda'_n \mu'_n)\rho')\end{aligned} \tag{11}$$

where $\lambda = \omega_1 - \omega_2$, $\mu = \omega_2 - \omega_3$ etc,

$$\begin{aligned}(n' \| a^\dagger \| n) = {} & \left(\frac{(n_1+4)(n_1-n_2+2)(n_1-n_3+3)}{(n_1-n_2+3)(n_1-n_3+4)} \right)^{1/2} \delta_{n'_1, n_1+2} \delta_{n'_2, n_2} \delta_{n'_3, n_3} \\ & + \left(\frac{(n_2+3)(n_1-n_2)(n_2-n_3+2)}{(n_1-n_2-1)(n_2-n_3+3)} \right)^{1/2} \delta_{n'_1, n_1} \delta_{n'_2, n_2+2}, \delta_{n'_3, n_3} \\ & + \left(\frac{(n_3+2)(n_2-n_3)(n_1-n_3+1)}{(n_1-n_3)(n_2-n_3+1)} \right)^{1/2} \delta_{n'_1, n_1} \delta_{n'_2, n_2} \delta_{n'_3, n_3+2}\end{aligned} \tag{12}$$

(cf. also Quesne 1981) and U is an SU(3) Racah coefficient (Draayer and Akiyama 1973). Λ is conveniently diagonal in this basis with eigenvalues, independent of ρ,

$$\Omega(\sigma n \omega) = (\sigma - 1) \sum_i n_i + \tfrac{1}{2}\left(\sum_i (\omega_i - \sigma)^2 + 2\omega_1 - 2\omega_3 \right) - \tfrac{1}{4}\left(\sum_i n_i^2 + 2n_1 - 2n_3 \right) \tag{13}$$

with $\sigma = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$. Thus, if $\Psi_{\sigma n \rho \omega}$ is the coherent state wavefunction for state $|\sigma n \rho \omega)$, then, from equation (4),

$$(\Psi_{\sigma n' \rho' \omega'} \| z \| \psi_{\sigma n \rho \omega}) = (\sigma n' \rho' \omega' \| a^\dagger \| \sigma n \rho \omega) \tag{14}$$

and hence

$$(\Psi_{\sigma n' \rho' \omega'} \| [\Lambda, z] \| \Psi_{\sigma n \rho \omega}) = (\Omega(\sigma n' \omega') - \Omega(\sigma n \omega))(\sigma n' \rho' \omega' \| a^\dagger \| \sigma n \rho \omega). \tag{15}$$

We now consider the RHS of equation (9). Since κ is a U(3) scalar, it has vanishing matrix elements between states of different ω. In general, states of given ω occur with multiplicity indexed by $n\rho$. It follows that any state ω that is multiplicity free must be an eigenstate of κ. Hence, if ω and ω' are both multiplicity free, we obtain from equations (9) and (15)

$$(\kappa^2(\omega')/\kappa^2(\omega))(\sigma n' \rho' \omega' \| a^\dagger \| \sigma n \rho \omega) = (\Omega(\sigma n' \omega') - \Omega(\sigma n \omega))(\sigma n' \rho' \omega' \| a^\dagger \| \sigma n \rho \omega). \tag{16}$$

From equation (10), it then follows that, for multiplicity free states,

$$(\sigma n'\rho'\omega'\|A\|\sigma n\rho\omega) = (\Omega(\sigma n'\omega') - \Omega(\sigma n\omega))^{1/2}(\sigma n'\rho'\omega'\|a^\dagger\|\sigma n\rho\omega), \qquad (17)$$

which is the desired analytic relationship.

Equation (17) gives all the matrix elements, consistent with Castaños *et al* (1983), for representations of the type ($\sigma_1 = \sigma_2 = \sigma_3$) since they contain only multiplicity free states. However, it also gives precisely the matrix elements between all multiplicity free states for arbitrary (σ_1, σ_2, σ_3).

Finally, we note that κ is also diagonal in the above basis for any (σ_1, σ_2, σ_3) in the large σ limit. If then we make the approximation of assuming that κ is diagonal, in general, we obtain equation (17) for all matrix elements.

Tables 1–4 give some matrix elements calculated with equation (17), both for states with and without multiplicity. They are compared with matrix elements calculated in the u(3)-boson limit and with exact elements computed numerically (Rosensteel and Rowe 1983). Tables 1–2 show results for the $N_\sigma(\lambda_\sigma\mu_\sigma) = 24.5(0, 4)$ representations, appropriate for a description of rotational bands in the light ^{12}C nucleus, while tables

Table 1. Some basis states for the representation $N_\sigma(\lambda_\sigma\mu_\sigma) = 24.5(0, 4)$.

Index	n	ω	Index	n	ω
1	(0, 0, 0)	(9.5, 9.5, 5.5)	4	(2, 0, 0)	(9.5, 9.5, 7.5)
2	(2, 0, 0)	(11.5, 9.5, 5.5)	7, 1	(2, 2, 0)	(11.5, 9.5, 7.5)
3	(2, 0, 0)	(10.5, 9.5, 6.5)	7, 2	(4, 0, 0)	(11.5, 9.5, 7.5)

Table 2. Approximate and exact sp(3, R) matrix elements for the 24.5(0, 4) representation.

Index		$\langle i\|A\|j\rangle$		
i	j	u(3)-boson	Approx	Exact
2	1	4.042	4.3589	4.3589
3	1	4.042	3.4641	3.4641
4	1	4.042	2.6458	2.6458
7, 1	2	3.810	2.6667	2.6638
7, 1	3	−3.300	−3.1623	−3.1615
7, 1	4	2.694	2.9814	2.9848
7, 2	2	1.6102	0.8909	0.8993
7, 2	3	4.4622	3.8247	3.8254
7, 2	4	3.1879	3.2523	3.2493

Table 3. Some basis states for the representation 733(82, 0).

Index	n	ω	Index	n	ω
13	(2, 2, 2)	(301, 219, 219)	32, 1	(4, 2, 2)	(301, 221, 219)
16	(4, 2, 0)	(301, 221, 217)	32, 2	(4, 4, 0)	(301, 221, 219)
18	(4, 2, 0)	(301, 220, 218)	32, 3	(6, 2, 0)	(301, 221, 219)

Table 4. Approximate and exact sp(3, R) matrix elements for the 733(82, 0) representation.

Index		$\langle i\|A\|j\rangle$		
i	j	u(3)-boson	Approx	Exact
32, 1	13	27.07	25.514 70	25.514 69
32, 1	16	21.52	20.236 69	20.236 67
32, 1	18	7.64	7.214 37	7.214 37
32, 2	13	0	0	−0.000 01
32, 2	16	13.13	12.306 76	12.306 78
32, 2	18	−23.30	−21.937 47	−21.937 47
32, 3	13	0	0	0.000 02
32, 3	16	1.27	1.181 77	1.181 78
32, 3	18	11.23	10.533 18	10.533 18

3–4 show results for the 733(82, 0) representation, used for the heavy ^{154}Sm nucleus. (Note that $N_\sigma = \sigma_1 + \sigma_2 + \sigma_3$.) It is seen that the (approximate) matrix elements obtained with the analytical formula are exact for multiplicity free states and remarkably accurate in general. For the ^{154}Sm representation, the approximation is accurate to ~2 parts in 10^6 and even for ^{12}C the results are accurate to a fraction of 1%. Note too that, for any representation, the 0 and $2\hbar\omega$ states are always multiplicity free. Thus the analytical formula is extremely useful and, for most practical applications, at least in nuclear physics, it is as good as exact.

It is shown in Rowe (1983) that equation (17) is accurate to terms of the order $[(\lambda_\sigma + \mu_\sigma)/2\sigma]^4$. It is also shown how exact results can be obtained when this level of accuracy is inadequate.

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